

# Theoretical design and circuit implementation of integer domain chaotic systems

Qianxue Wang, Simin Yu, Christophe Guyeux,  
Jacques Bahi, and Xiaole Fang

August 23, 2016

## Abstract

In this paper, a new approach for constructing integer domain chaotic systems (IDCS) is proposed, and its chaotic behavior is mathematically proven according to the Devaney's definition of chaos. Furthermore, an analog-digital hybrid circuit is also developed for realizing the designed basic IDCS. In the IDCS circuit design, chaos generation strategy is realized through a sample-hold circuit and a decoder circuit so as to convert the uniform noise signal into a random sequence, which plays a key role in circuit implementation. The experimental observations further validate the proposed systematic methodology for the first time.

## 1 Introduction

Currently, international research works on chaotic systems and their applications are mainly concentrated on real domain, leading to the study and development of the so called Real Domain Chaotic Systems (RDCSs). RDCSs are divided in two categories: continuous-time and discrete-time chaotic systems. Continuous-time systems, including the Lorenz, Chen and Chua systems [14, 8, 7] are defined by state (differential) equations, while discrete time chaotic systems are given by iterative equations, such as Logistic or Henon map, Chen-Lai algorithm [15, 17, 6], and so on.

When a chaotic phenomenon is implemented either in digital computers or in some other digital devices, its associated chaotic system is discretized both spatially and temporally. That is, the system becomes both a discrete-time and discrete-valued pseudo chaotic system, defined in discrete time and on finite spatial lattice [9]. Finite words lengths in the digital machines lead to finite precision effects, and may result in consequent dynamic degradation, such as short cycle-length, non-ideal distribution and correlation, low linear complexity, and so on [12]. Since the early implementation of such systems in finite state machines, many researchers have done several improvements to overcome the problems arising from the digitized chaotic systems, like using higher finite precision [13], perturbation-based algorithms to effectively compensate the dynamics degradation [16], and cascading multiple chaotic systems to obtain a greater period [11], but they have not achieved to solve the problem fundamentally.

Integer Domain Chaotic Systems (IDCSs), for their part, refer to chaotic systems defined on an integer domain. Their main feature is essentially to solve the problem of dynamics degradation caused by finite precision effects. In 2010, the research team at the University of Franche-Comté (France) has proposed a new IDCS designated as CI (Chaotic Iterations) system [4]. This CI system uses only bitwise operations, thus achieving the speed requirement. Furthermore, theoretical analyses show that these CI systems on integer domains satisfy the Devaney's definition of chaos. Since these systems run on finite sets of integer domains, then the finite precision problem disappears, and there is no need of any transformation from real numbers to binary sequences. CI system is one of effective solutions for the aforementioned problems that occur in the RDSCS case.

The first collaborative work has consisted in chaotically combine two random inputs in order to construct a first CI system, called PRIM CI in [1], which has led to better statistical properties for the resulted pseudorandom number generator than each input taken alone. A second category of CI systems called MARK CI has then been introduced in [2]: a mark sequence has been applied to avoid wasteful duplication of values, leading by doing so to an obvious speed improvement. The LUT (Lookup-Table) CI has finally be released and deeply studied in [3]: this last version of the chaotic combination of two input entropic streams has solved flaws exhibited by the MARK CI version of these IDCSs.

In this article, a novel approach for generating IDCS and its proof of the existence of chaos according to Devaney's definition is presented. We focus on the design and circuit implementation of IDCS, with theoretical background and practical details presented for the first time. The IDCS circuit design consists of uniform noise signal generator, noise voltage converter, sample and hold circuit, decoder circuit, iterate function circuit, and digital to analog converter six parts together. The main feature of this kind of IDCS circuit is the use of a sample-hold circuit and a decoder circuit to convert the uniform noise signal into a random sequence, which plays a key role in generating IDCS signals.

The remainder of this research work is organized as follows. The description of IDCS is given in Section 2, while the proof of chaos is provided in Section 3. Circuit design and implementation of IDCS are detailed in the next section. This article ends by a conclusion section in which the contributions are summarized (Section 5).

## 2 Description of IDCS

In this section, we first introduce the basic concept of IDCSs.

### 2.1 RDSCS

In the traditional RDSCS studies, the general form of the iterative equations is:

$$x_0 \in \mathbb{R}, \text{ and } \forall n \in \mathbb{N}^*, x_n = f(x_{n-1}),$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is the iteration function, while  $x_{n-1}$  and  $x_n$  are the  $n - 1$ -th and  $n$ -th iteration respectively. Note that  $x_{n-1}$  and  $x_n$  are real numbers, which are represented in

binary form as

$$\begin{cases} x_{n-1} = (x_{i_1} x_{i_2} \dots x_{i_M} x_{j_1} x_{j_2} \dots x_{j_N} \dots)_2 \\ x_n = (x_{k_1} x_{k_2} \dots x_{k_L} x_{l_1} x_{l_2} \dots x_{l_P} \dots)_2 \end{cases} \quad (1)$$

where  $x_{i_1}, x_{i_2}, \dots, x_{i_M} \in \{0, 1\}$  and  $x_{j_1}, x_{j_2}, \dots, x_{j_N}, \dots \in \{0, 1\}$  are respectively the integer and fractional parts for  $x_{n-1}$ . Similarly,  $x_{k_1}, x_{k_2}, \dots, x_{k_L} \in \{0, 1\}$  and  $x_{l_1}, x_{l_2}, \dots, x_{l_P}, \dots \in \{0, 1\}$  are respectively the integer and fractional parts for  $x_n$ .

The main features of discrete-time RDCS is that all the bits in  $x_{n-1}$  will be updated by iteration function  $f$  at each operation (iteration). Likewise, all the bits in  $x_n$  will be updated by iteration function  $f$  at each operation (iteration).

## 2.2 IDCS

The main ideas of CI systems are summarized thereafter.

Let  $N \in \{1, 2, \dots\}$  be a positive integer,  $\mathbb{B} = \{1, 0\}$  denotes the set of binary numbers, while  $\mathbb{B}^N$  is the set of binary vectors of size  $N$ . For any  $n = 0, 1, 2, \dots$ ,  $x^n$  is represented by using  $N$  bits in base-2:  $x^0 = (x_{N-1}^0 x_{N-2}^0 \dots x_0^0) \in \mathbb{B}^N$  is the initial condition, while  $x^{n-1} = (x_{N-1}^{n-1} x_{N-2}^{n-1} \dots x_0^{n-1}) \in \mathbb{B}^N$  and  $x^n = (x_{N-1}^n x_{N-2}^n \dots x_0^n) \in \mathbb{B}^N$  denote the  $n-1$ -th and  $n$ -th iteration respectively. In CI systems, the iterative equation is defined as follows:

$$x_i^n = \begin{cases} x_i^{n-1} & \text{if } i \neq s^n \\ (f(x^{n-1}))_i & \text{if } i = s^n, \end{cases} \quad (2)$$

where  $i = 0, 1, 2, \dots, N-1$ ,  $n = 1, 2, \dots$ , and  $s = (s^1 s^2 \dots s^n \dots)$  is an one-sided infinite sequence of integers bounded by  $N-1$ :  $\forall n \in \mathbb{N}^*$ ,  $s^n \in \{0, 1, 2, \dots, N-1\}$ . Additionally, the iterate function  $f$  is usually the vectorial Boolean negation, given by  $f(x^{n-1}) = (\overline{x_{N-1}^{n-1}} \overline{x_{N-2}^{n-1}} \dots \overline{x_i^{n-1}} \dots \overline{x_0^{n-1}})$ , and the following notation is used:  $(f(x^{n-1}))_{i=s^n} = (\overline{x_{N-1}^{n-1}} \overline{x_{N-2}^{n-1}} \dots \overline{x_i^{n-1}} \dots \overline{x_0^{n-1}})_{i=s^n} = \overline{x_{i=s^n}^{n-1}}$ , that is,  $(f(x^{n-1}))_{i=s^n}$  is the  $i$ -th component of  $f(x^{n-1})$ . Let us finally remark that, in IDCS, the one-sided infinite sequence of integers  $s = (s^1 s^2 \dots s^n \dots)$  is usually named a chaotic strategy.

Let  $x_k, x_j$  be two binary digits, the corresponding distance be

$$\delta(x_j, x_k) = \begin{cases} 1 & \text{if } x_j \neq x_k \\ 0 & \text{if } x_j = x_k. \end{cases} \quad (3)$$

Using the same notations as above, we define the binary variables negation as follows:

$$(F_f(k, x))_j = x_j \cdot \delta(k, j) + (f(x))_k \cdot \overline{\delta(k, j)}, \quad (4)$$

where  $j \in \{0, 1, 2, \dots, N-1\}$ , and  $k$  is usually a term of chaotic strategy  $s$  while  $f$  is often the vectorial negation recalled previously. With these choices, and according to Equation 4, a more specific formula can be obtained:

$$F_f(k, x) = (x_{N-1}, x_{N-2}, \dots, x_{k+1}, \overline{x_k}, x_{k-1}, \dots, x_1, x_0).$$

Let  $E = (s, x)$  be a couple constituted by a chaotic strategy and a Boolean vector, that is,  $E = (s, x) \in \mathcal{E} = \{0, 1, 2, \dots, N-1\}^\infty \times \mathbb{B}^N$ . Define function  $G_f$  as follows:

$$G_f(E) = G_f((s, x)) = (\sigma(s), F_f(i(s), x)), \quad (5)$$

where  $i(s) = s^1$  and  $\sigma^k(s) = \underbrace{\sigma \circ \sigma \circ \dots \circ \sigma}_k(s)$ ,  $k = 1, 2, \dots$  is the result for shifting  $k$  integers in the one-sided infinite sequence  $s = (s^1 s^2 \dots s^n \dots)$  to the left. In other words,

$$\sigma^k(s) = s^{k+1} s^{k+2} \dots s^n \dots (k = 1, 2, \dots).$$

With all this material, IDCS is defined as follows:

$$E^0 \in \mathcal{E} \text{ and } \forall k \in \mathbb{N}, E^{k+1} = G_f(E^k).$$

Consider now two real numbers  $a$  and  $b$ , lesser than 1, which are represented in radix- $r$  format as

$$\begin{cases} a = 0.a_1 a_2 a_3 \dots a_n \dots & = \sum_{k=1}^{\infty} \frac{a_k}{r^k} \\ b = 0.b_1 b_2 b_3 \dots b_n \dots & = \sum_{k=1}^{\infty} \frac{b_k}{r^k}, \end{cases} \quad (6)$$

where  $a_k, b_k \in \{0, 1, 2, \dots, r-1\}$ . Then the distance between  $a$  and  $b$  is given by:

$$d(a, b) = \sum_{k=1}^{\infty} \frac{|a_k - b_k|}{r^k} \quad (7)$$

The above formula can be generalized to calculate the distance between two one-sided infinite sequences of symbols without loss of generality. These remarks lead to the definition of a new distance on the set  $\mathcal{E}$ , which is defined by:

$$d((s, x), (\hat{s}, \hat{x})) = d_s(s, \hat{s}) + d_x(x, \hat{x}),$$

where  $s = (s^1 s^2 \dots s^n \dots)$  and  $\hat{s} = (\hat{s}^1 \hat{s}^2 \dots \hat{s}^n \dots)$  are one-sided infinite sequences of integers, while  $x$  and  $\hat{x}$  are binary digits of  $N$  bits. More precisely, and in agreement with Equation 7, the distance between  $s$  and  $\hat{s}$  is:

$$d_s(s, \hat{s}) = \sum_{k=1}^{\infty} \frac{|s^k - \hat{s}^k|}{N^k} \in [0, 1] \quad (8)$$

where  $\forall k \in \mathbb{N}^*$ ,  $s^k, \hat{s}^k \in \{0, 1, 2, \dots, N-1\}$ . Finally, following the 1-norm distance, the distance between  $x$  and  $\hat{x}$  is:

$$d_x(x, \hat{x}) = \sum_{k=0}^{N-1} \delta(x_k, \hat{x}_k) \in \{0, 1, 2, \dots, N\}. \quad (9)$$

Remark that  $d$  is a distance, as it is defined as the sum of two distances.

Before investigating the chaotic properties of IDCS, we have to prove that  $G_f$  is continuous on the metric space  $(\mathcal{E}, d)$ . In order to do so, the following lemma is first introduced:

**Lemma 1** *Let  $s = (s^1 s^2 \dots s^n \dots)$  and  $\hat{s} = (\hat{s}^1 \hat{s}^2 \dots \hat{s}^n \dots)$ , where  $s^k, \hat{s}^k \in \{0, 1, 2, \dots, N-1\}$  for  $k = 1, 2, \dots$ . If  $s^i = \hat{s}^i$  for  $i = 1, 2, \dots, n$ , then  $d(s, \hat{s}) \leq \frac{1}{N^n}$ . Conversely, if  $d(s, \hat{s}) \leq \frac{1}{N^n}$ , then  $s^i = \hat{s}^i$  for  $i = 1, 2, \dots, n$ .*

PROOF If  $s^i = \hat{s}^i$  ( $i = 1, 2, \dots, n$ ), then

$$\begin{aligned} d(s, \hat{s}) &= \sum_{i=1}^n \frac{|s^i - \hat{s}^i|}{N^i} + \sum_{i=n+1}^{\infty} \frac{|s^i - \hat{s}^i|}{N^i} = \sum_{i=n+1}^{\infty} \frac{|s^i - \hat{s}^i|}{N^i} \\ &\leq \sum_{i=n+1}^{\infty} \frac{N-1}{N^i} = (N-1) \frac{\frac{1}{N^{n+1}}}{1 - \frac{1}{N}} = \frac{1}{N^n} \end{aligned}$$

Conversely, and due to the definition of the proposed distance: for any  $m \leq n$ , if  $s^m \neq \hat{s}^m$ , then  $d(s, \hat{s}) \geq \frac{1}{N^m}$ . The contraposition is the desired result: if  $d(s, \hat{s}) \leq \frac{1}{N^n}$ , then  $s^i = \hat{s}^i$  ( $i = 1, 2, \dots, n$ ).

To prove that chaotic iterations are an example of chaos, we must first set that  $G_f$  is continuous on the metric space  $(\mathcal{E}, d)$ .

**Theorem 1**  $G_f$  is a continuous function.

PROOF A continuous function is a function for which, intuitively, "small" changes in the input result in "small" changes in the output. Let  $((s, x)_n)_{n \in \mathbb{N}}$  be a sequence of the phase space  $\mathcal{E}$ , which converges to  $(\hat{s}, \hat{x})$ . We will prove that  $(G_f(s, x)_n)_{n \in \mathbb{N}}$  converges to  $G_f(\hat{s}, \hat{x})$ . In mathematical notation,  $\forall ((s, x)_n)_{n \in \mathbb{N}} \subset \mathcal{E} : \lim_{n \rightarrow \infty} (s, x)_n = (\hat{s}, \hat{x}) \Rightarrow \lim_{n \rightarrow \infty} G_f((s, x)_n) = G_f(\hat{s}, \hat{x})$

1.  $\lim_{n \rightarrow \infty} (s, x)_n = (\hat{s}, \hat{x}) \Rightarrow d((s, x)_n, (\hat{s}, \hat{x})) < \delta$

Without loss of generality, we assume that  $\delta < 1$ .

2. If  $(x)_n \neq \hat{x}$ , then  $d_x((x)_n, \hat{x}) \geq 1$ , and so  $d((s, x)_n, (\hat{s}, \hat{x})) = d_s((s)_n, \hat{s}) + d_x((x)_n, \hat{x}) > \delta$ . Thus  $\exists n_0 \in \mathbb{N}, d_x((x)_n, \hat{x}) = 0$  for any  $n \geq n_0$
3. As  $d((s, x)_n, (\hat{s}, \hat{x})) < \delta$  making

$$d((s, x)_n, (\hat{s}, \hat{x})) = d_s((s)_n, \hat{s}) + d_x((x)_n, \hat{x}) = d_s((s)_n, \hat{s}) < \delta$$

According to the previous Lemma 1, if the  $k_0$  first elements of  $(s)_n$  and  $\hat{s}$  are the same, then  $d_s(\hat{s}, \tilde{s}) < N^{-k_0} < \delta$ . For instance,  $k_0 = \text{floor}(-\log_N \delta) + 1$  is convenient. Thus  $\exists n_1 \in \mathbb{N}, d_s((s)_n, \hat{s}) < \delta$  for any  $n \geq n_1$

4. According to Equation 5, the corresponding  $G_f((s, x)_n)$  and  $G_f(\hat{s}, \hat{x})$  can be obtained:

$$G_f((s, x)_n) = (\sigma((s)_n), F_f(i((s)_n), (x)_n))$$

$$G_f(\hat{s}, \hat{x}) = (\sigma(\hat{s}), F_f(i(\hat{s}), \hat{x}))$$

For  $n \geq \max(n_0, n_1)$ , the  $k_0$  first elements of  $(s)_n$  and  $\hat{s}$  are the same and  $(x)_n = \hat{x}$ , so

$$i((s)_n) = i(\hat{s})$$

Then,

$$F_f(i((s)_n), (x)_n) = F_f(i(\hat{s}), \hat{x})$$

$\sigma(s)$  is the result for shifting one integer in the one-sided infinite sequence to the left. So the  $k_0 - 1$  first elements of  $\sigma((s)_n)$  and  $\sigma(\hat{s})$  are still the same.

So

$$d(G_f((s, x)_n), G_f(\hat{s}, \hat{x})) = d((\sigma((s)_n), \sigma(\hat{s}) + d(F_f(i((s)_n), (x)_n), F_f(i(\hat{s}), \hat{x}))) = d((\sigma((s)_n), \sigma(\hat{s})) < N^{-(k_0-1)}$$

make

$$\lim_{n \rightarrow \infty} G_f((s, x)_n) = G_f(\hat{s}, \hat{x})$$

true.

In conclusion,

$G_f$  is consequently continuous.

### 3 Proof of chaos for IDCS

In this section, the chaotic behavior of IDCS is proven according to the Devaney's definition recalled below.

**Definition 1 (Devaney's definition of chaos [10])** Let  $f : X \rightarrow X$  be a continuous function on the metrical space  $(X, d)$ . The dynamical system  $x_0 \in X, x_{n+1} = f(x_n)$  is said chaotic if:

1. its periodic points are dense in  $X$ ;
2. it is transitive;
3. it has sensitive dependence on initial conditions.

The meaning of these properties are detailed thereafter. Let us recall before that.

**Theorem 2 [5]** *If a dynamical system is transitive and has dense periodic points, then it has sensitive dependence on initial conditions.*

#### 3.1 Dense periodic points

**Theorem 3** *The periodic points of  $G_f$  are dense in  $\mathcal{E}$ .*

**PROOF** We want to show that, for any given  $\varepsilon > 0$ , a periodic point  $(\tilde{s}, \tilde{x}) \in \mathcal{E}$  can always be found within range  $\varepsilon$  of any point  $(\hat{s}, \hat{x}) \in \mathcal{E}$ .

1. Without loss of generality, we assume that the given  $\varepsilon < 1$  and that the general form of  $(\hat{s}, \hat{x})$  is

$$(\hat{s}, \hat{x}) = ((s^1 s^2 \dots s^{k_0} \dots s^n \dots), \hat{x}) \in \mathcal{E}$$

2. If  $\tilde{x} \neq \hat{x}$ , then  $d_x(\tilde{x}, \hat{x}) \geq 1$ , and so  $d((\hat{s}, \hat{x}), (\tilde{s}, \tilde{x})) > 1$ . Thus  $\tilde{x} = \hat{x}$ .

3. If the  $k_0$  first elements of  $\hat{s}$  and  $\tilde{s}$  are the same, then  $d_s(\hat{s}, \tilde{s}) < N^{-k_0}$  according to the previous Lemma 1. So,  $\forall \varepsilon < 1$ , an integer  $k_0$  can always be found making the relation  $d_s(\hat{s}, \tilde{s}) < N^{-k_0} < \varepsilon$  true. For instance,  $k_0 = \text{floor}(-\log_N \varepsilon) + 1$  is convenient.
4. If after  $k_0$ -th iteration, we have

$$\tilde{x} = \hat{x} = G_f^{k_0}((\tilde{s}, \tilde{x}))_2,$$

then a cycle point  $(\tilde{s}, \tilde{x}) = ((s^1 s^2 \dots s^{k_0} s^1 s^2 \dots s^{k_0} \dots), \tilde{x}) \in \mathcal{E}$  is found that satisfies

$$(\tilde{s}, \tilde{x}) = G_f^{k_0}((\tilde{s}, \tilde{x})),$$

making

$$d((\hat{s}, \hat{x}), (\tilde{s}, \tilde{x})) = d_s(\hat{s}, \tilde{s}) + d_x(\hat{x}, \tilde{x}) = d_s(\hat{s}, \tilde{s}) < \varepsilon$$

true.

5. If after  $k_0$ -th iteration, we have

$$\tilde{x} \neq \hat{x} = G_f^{k_0}((\tilde{s}, \tilde{x}))_2$$

Then, without loss of generality, we can assume that there are  $i_0$  ( $\leq N$ ) different bits between  $\hat{x}$  and  $\tilde{x}$ . These  $i_0$  bits are numbered  $j_1 < j_2 < \dots < j_{i_0}$  respectively. To obtain that, after another  $k_0$  iterations, the following condition is met

$$\tilde{x} = \hat{x} = G_f^{k_0+i_0}((\tilde{s}, \tilde{x}))_2,$$

we must set:

$$\begin{cases} s^{k_0+1} &= j_1 \\ s^{k_0+2} &= j_2 \\ \dots & \\ s^{k_0+i_0} &= j_{i_0}. \end{cases} \quad (10)$$

Then, within range  $\varepsilon$  of the point  $(\hat{s}, \hat{x})$ , one can find the following periodic point

$$(\tilde{s}, \tilde{x}) = ((s^1 s^2 \dots s^{k_0} s^{k_0+1} s^{k_0+2} \dots s^{k_0+i_0} s^1 s^2 \dots s^{k_0} s^{k_0+1} s^{k_0+2} \dots s^{k_0+i_0} \dots), \tilde{x}) = G_f^{k_0+i_0}(\tilde{s}, \tilde{x}) \in \mathcal{E}$$

making

$$d((\hat{s}, \hat{x}), (\tilde{s}, \tilde{x})) = d_s(\hat{s}, \tilde{s}) + d_x(\hat{x}, \tilde{x}) = d_s(\hat{s}, \tilde{s}) < \varepsilon$$

true.

In summary, the periodic points of  $G_f$  are dense in  $\mathcal{E}$ .

### 3.2 Transitive property

**Theorem 4**  $G_f$  is a transitive map on  $\mathcal{E}$ .

PROOF The so-called topological transitivity specifically refers to that: for any nonempty open sets  $U_A$  and  $U_B$  in  $(\mathcal{E}, d)$ , there is always  $n_0 > 0$  that makes  $G_f^{n_0}(U_A) \cap U_B \neq \emptyset$ .

Consider now two nonempty open sets  $U_A$  and  $U_B$ , and  $(s_A, x_A) \in U_A$ ,  $(s_B, x_B) \in U_B$ .  $U_A$  and  $U_B$  are open, and we take place in a metric space, so there exist real numbers  $r_A > 0$  and  $r_B > 0$  such that the open ball  $\mathcal{B}_A$  of center  $(s_A, x_A)$  and radius  $r_A$  is inside  $U_A$  (resp. the open ball  $\mathcal{B}_B$  of center  $(s_B, x_B)$  and radius  $r_B$  is into  $U_B$ ). Without loss of generality, we can suppose that  $r_A < 1$ .

1. We introduce the following notations:

$$(s_A, x_A) = ((s_A^1 s_A^2 \dots s_A^{n_0} \dots s_A^n \dots), x_A) \in U_A \subseteq \mathcal{E}$$

and

$$(s_B, x_B) = ((s_B^1 s_B^2 \dots s_B^n \dots), x_B) \in U_B \subseteq \mathcal{E}.$$

2. Let  $(\tilde{s}, \tilde{x}) \in U_A$ . If  $\tilde{x} \neq x_A$ , then  $d_x(\tilde{x}, x_A) \geq 1$ , and so  $d((s_A, x_A), (\tilde{s}, \tilde{x})) > 1$ . Consequently, if  $(\tilde{s}, \tilde{x}) \in \mathcal{B}_A$ , then  $d((s_A, x_A), (\tilde{s}, \tilde{x})) < r_A < 1$ , and so  $\tilde{x} = x_A$ .
3. If we demand that the  $k_0$  first elements of  $\tilde{s}$  are equal to those from  $s_A$ , then we obtain  $d_s(s_A, \tilde{s}) < N^{-k_0}$ . And for the given  $r_A$ , an integer  $k_0$  (that is, a sequence  $\tilde{s}$ ) can always be found to achieve  $d_s(s_A, \tilde{s}) < N^{-k_0} < r_A$  (for instance,  $k_0 = \text{floor}(-\log_N r_A) + 1$ ).
4. If after  $k_0$  iterations, the following condition is satisfied:

$$G_f^{k_0}((s_A, x_A))_2 = x_B,$$

then  $n_0 = k_0$  and  $(\tilde{s}, \tilde{x}) = ((s_A^1 s_A^2 \dots s_A^{n_0} s_B^1 s_B^2 \dots s_B^n \dots), x_A) \in U_A$  has been found that satisfy:

$$G_f^{n_0}(\tilde{s}, \tilde{x}) = (s_B, x_B) \in G_f^{n_0}(U_A) \cap U_B,$$

making

$$G_f^{n_0}(U_A) \cap U_B \neq \emptyset$$

true.

5. If, after the  $k_0$ -th iteration,

$$G_f^{k_0}((s_A, x_A))_2 \neq x_B,$$

then, without loss of generality, we can assume that there are  $i_0 (\leq N)$  different bits between  $x_B$  and the Boolean vector of  $G_f^{k_0}((s_A, x_A))$ . Once again, these  $i_0$  bits are numbered  $j_1 < j_2 < \dots < j_{i_0}$  respectively. Define now,

$$\begin{cases} s^{k_0+1} &= j_1 \\ s^{k_0+2} &= j_2 \\ \dots & \\ s^{k_0+i_0} &= j_{i_0} \end{cases} \quad (11)$$



so the point  $(\tilde{s}, \tilde{x}) = ((s_A^1 s_A^2 \dots s_A^{k_0} j_1 j_2 \dots j_{i_0} s_B^1 s_B^2 \dots s_B^n \dots), x_A) \in U_A$  satisfies

$$G_f^{n_0}(\tilde{s}, \tilde{x}) = (s_B, x_B) \in G_f^{n_0}(U_A) \cap U_B$$

with  $n_0 = k_0 + i_0$ , making the claim

$$G_f^{n_0}(U_A) \cap U_B \neq \emptyset$$

true.

In summary,  $(G_f, \mathcal{E})$  is transitive, as shown in Figure 1.

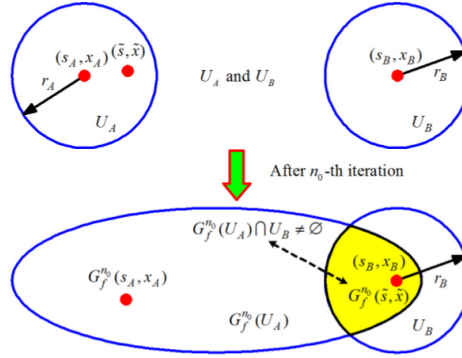


Figure 1: The schematic diagram of transitivity in  $(G_f, \mathcal{E})$

Because of dense periodic points and transitive, according to Definition 1 and Theorem 2, IDCS is chaotic in the sense of Devaney.

## 4 Circuit Implementation of IDCS

In this section, IDCS circuit is designed, which consists of several sub-modules: uniform noise signal generator, noise voltage converter, sample-hold circuit, decoder circuit, iterate function circuit, and digital to analog converter. Finally, both the validity and practicability are verified by the experimental results.

The uniform noise signal generator is provided in Figure 2. It uses MM5837 broadband white-noise generator with 3dB per octave filter from 10Hz to 40kHz to give noise output  $\xi(t)$ , which has flat spectral distribution over entire audio band from 20Hz to 20kHz. Output is about 1V<sub>P-P</sub> of noise riding on 8.5V level. The parameters of components in Figure 2 are capacitances  $C_1 = 100\mu F$ ,  $C_2 = 1\mu F$ ,  $C_3 = 0.27\mu F$ ,  $C_4 = C_5 = 0.047\mu F$ ,  $C_1 = 0.033\mu F$ , and resistances  $R_1 = 6.8k\Omega$ ,  $R_2 = 3k\Omega$ ,  $R_3 = 1k\Omega$ , and  $R_4 = 300\Omega$ . In Figure 2, output of uniform noise signal generator is about 1V<sub>P-P</sub> of noise riding on 8.5V level, so it should be converted to 0 ~ 4V uniform noise signal. Noise voltage converter is shown in Figure 3. The values of each resistance in Figure 3 are  $R_5 = R_6 = R_8 = R_9 = 10k\Omega$ ,  $R_7 = 40k\Omega$ . The noise output  $\xi(t) = 0 \sim 4V$ .

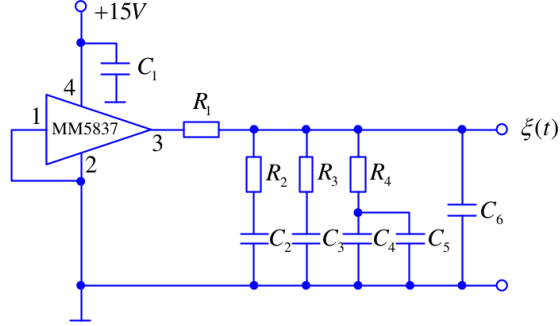


Figure 2: The uniform noise signal generator

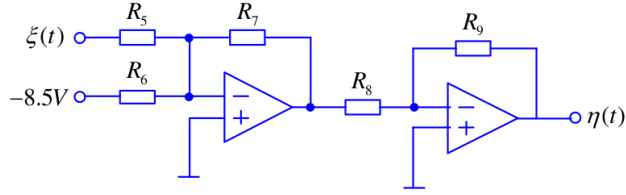


Figure 3: The noise voltage converter

Sample-and-hold circuit is shown in Figure 4, in which the chip model is LF398. Supply voltage is  $V_+ = +15V$ ,  $V_- = -15V$ . In Figure 4, 3-pin is for analog signal input, 5-pin is an output; capacitor  $C_F = 0.01 \sim 0.1\mu F$  ( $0.022\mu F$  is used here).  $u_c$  is a square wave signal with frequency  $1 \sim 5kHz$  ( $4kHz$  is used here), the amplitude of output is  $-5V \sim 5V$ . Notice that when  $C_F$  is enlarged, then the frequency of  $u_c$  is reduced,

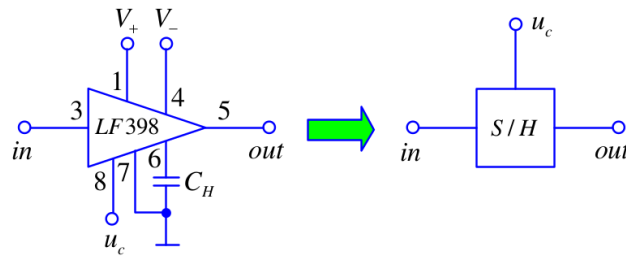


Figure 4: The sample-and-hold circuit

and so the iterations are more slower. Conversely, if  $C_F$  is smaller, then the frequency of  $u_c$  may be higher, and so the speed of iteration is faster. Due to the speed of the device itself, the speed of iteration has an upper limitation. When doing experiments,  $C_F$  should be a suitable value, the same for the frequency of  $u_c$ . That prevents work

abnormality. Decoding circuit is shown in Figure 5, while the corresponding comparator circuit is described in Figure 6. The values for each resistance are  $R_{10} = 13.5k\Omega$ ,  $R_{11} = 1k\Omega$ ,  $R_{12} = 10k\Omega$ ,  $R_{13} = 40\Omega$ , and  $R_{14} = R_{15} = R_{16} = 10k\Omega$ , while the voltage for inverting voltage shifter is  $E = 4V$ . According to Figure 6, the logical relationship for input and output of the comparator is as follows:

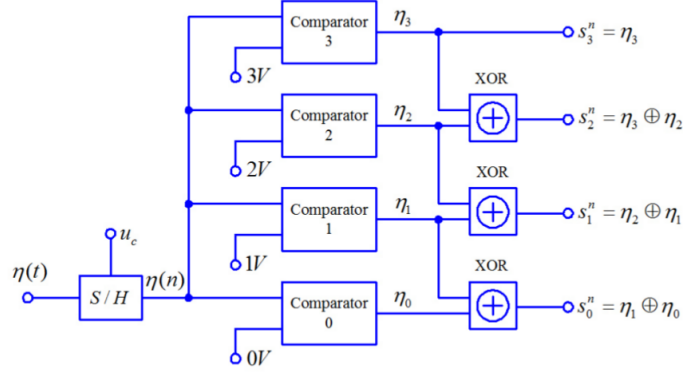


Figure 5: The decoder circuit

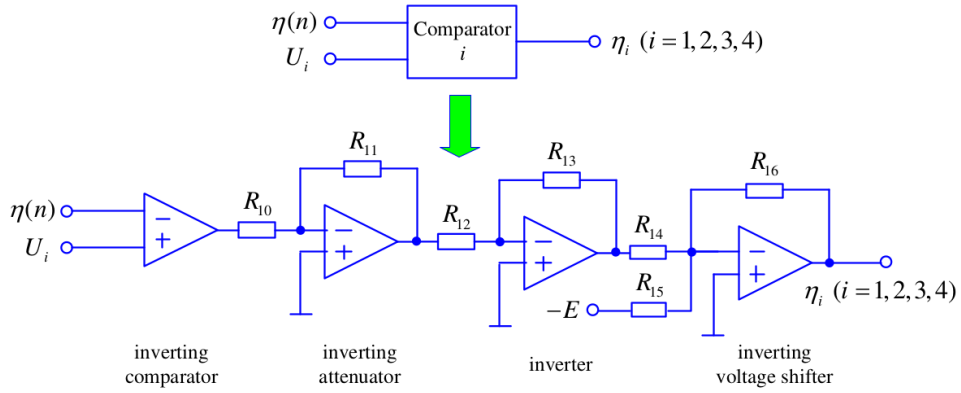


Figure 6: The comparator

$$\begin{cases} \text{if } \eta(n) > U_i, & \text{then } \eta_i = 1 \text{ (4V),} \\ \text{if } \eta(n) < U_i, & \text{then } \eta_i = 0 \text{ (0V).} \end{cases} \quad (12)$$

According to Figure 5, input-output relationship of the decoding circuit is:

1. When  $3V < \eta(t) \leq 4V$ , then  $\eta_3 = \eta_2 = \eta_1 = \eta_0 = 1$ , so

$$\begin{cases} s_3^n = \eta_3 = 1 \\ s_2^n = \eta_3 \oplus \eta_2 = 1 \oplus 1 = 0 \\ s_1^n = \eta_2 \oplus \eta_1 = 1 \oplus 1 = 0 \\ s_0^n = \eta_1 \oplus \eta_0 = 1 \oplus 1 = 0 \end{cases} \quad (13)$$

2. When  $2V < \eta(t) \leq 3V$ , then  $\eta_3 = 0, \eta_2 = \eta_1 = \eta_0 = 1$ , so

$$\begin{cases} s_3^n = \eta_3 = 0 \\ s_2^n = \eta_3 \oplus \eta_2 = 0 \oplus 1 = 1 \\ s_1^n = \eta_2 \oplus \eta_1 = 1 \oplus 1 = 0 \\ s_0^n = \eta_1 \oplus \eta_0 = 1 \oplus 1 = 0 \end{cases} \quad (14)$$

3. When  $1V < \eta(t) \leq 2V$ , then  $\eta_3 = \eta_2 = 0, \eta_1 = \eta_0 = 1$ , so

$$\begin{cases} s_3^n = \eta_3 = 0 \\ s_2^n = \eta_3 \oplus \eta_2 = 0 \oplus 0 = 0 \\ s_1^n = \eta_2 \oplus \eta_1 = 0 \oplus 1 = 1 \\ s_0^n = \eta_1 \oplus \eta_0 = 1 \oplus 1 = 0 \end{cases} \quad (15)$$

4. When  $0V < \eta(t) \leq 1V$ , then  $\eta_3 = \eta_2 = \eta_1 = 0, \eta_0 = 1$ , so

$$\begin{cases} s_3^n = \eta_3 = 0 \\ s_2^n = \eta_3 \oplus \eta_2 = 0 \oplus 0 = 0 \\ s_1^n = \eta_2 \oplus \eta_1 = 0 \oplus 0 = 0 \\ s_0^n = \eta_1 \oplus \eta_0 = 0 \oplus 1 = 1 \end{cases} \quad (16)$$

It can be seen in Figure 3 that the noise output satisfies  $0V < \eta(t) \leq 4V$ , and  $\eta(t)$  is the random signal that follows an equal probability distribution (i.e., uniform distribution) within the range  $[0V, 4V]$ . In other words, the values in these four intervals ( $[0V, 1V], [1V, 2V], [2V, 3V], [3V, 4V]$ ) are uniformly distributed, and the correspondence relationship between the size of  $s^n$  and the four intervals is:

$$\begin{cases} \text{if } \eta(t) \in [0V, 1V], & \text{then } s^n = 0 \\ \text{if } \eta(t) \in [1V, 2V], & \text{then } s^n = 1 \\ \text{if } \eta(t) \in [2V, 3V], & \text{then } s^n = 2 \\ \text{if } \eta(t) \in [3V, 4V], & \text{then } s^n = 3 \end{cases} \quad (17)$$

Through the above comparison, it is known that both  $s^n$  and  $s_3^n s_2^n s_1^n s_0^n$  follow the uniform distribution, the relationship between them satisfies:

$$\begin{cases} \text{if } \eta(t) \in [0V, 1V], & \text{then } s^n = 0 \Leftrightarrow s_3^n s_2^n s_1^n s_0^n = 0001 \\ \text{if } \eta(t) \in [1V, 2V], & \text{then } s^n = 1 \Leftrightarrow s_3^n s_2^n s_1^n s_0^n = 0010 \\ \text{if } \eta(t) \in [2V, 3V], & \text{then } s^n = 2 \Leftrightarrow s_3^n s_2^n s_1^n s_0^n = 0100 \\ \text{if } \eta(t) \in [3V, 4V], & \text{then } s^n = 3 \Leftrightarrow s_3^n s_2^n s_1^n s_0^n = 1000 \end{cases} \quad (18)$$

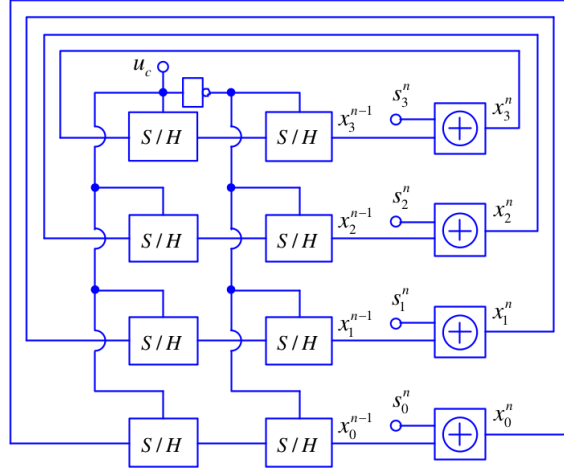


Figure 7: The iterate function circuit

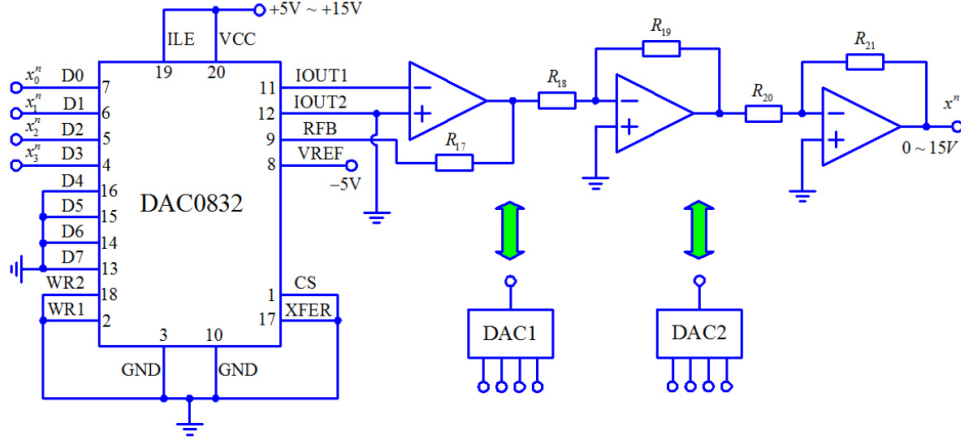


Figure 8: The digital to analog converter

Set  $N = 4$ , get basic iterative function for IDCS,

$$x_i^n = \begin{cases} x_i^{n-1} & \text{if } i \neq s^n \\ (f(x^{n-1}))_i = \overline{x_i^{n-1}} & \text{if } i = s^n \end{cases} \quad (19)$$

where  $s^n \in \{0, 1, 2, \dots, N-1\} = \{0, 1, 2, 3\}$  and  $i = 0, 1, 2, 3$ . By comparing Equation 13 to Equation 19, Equation 19 is equivalent to another type of mathematical

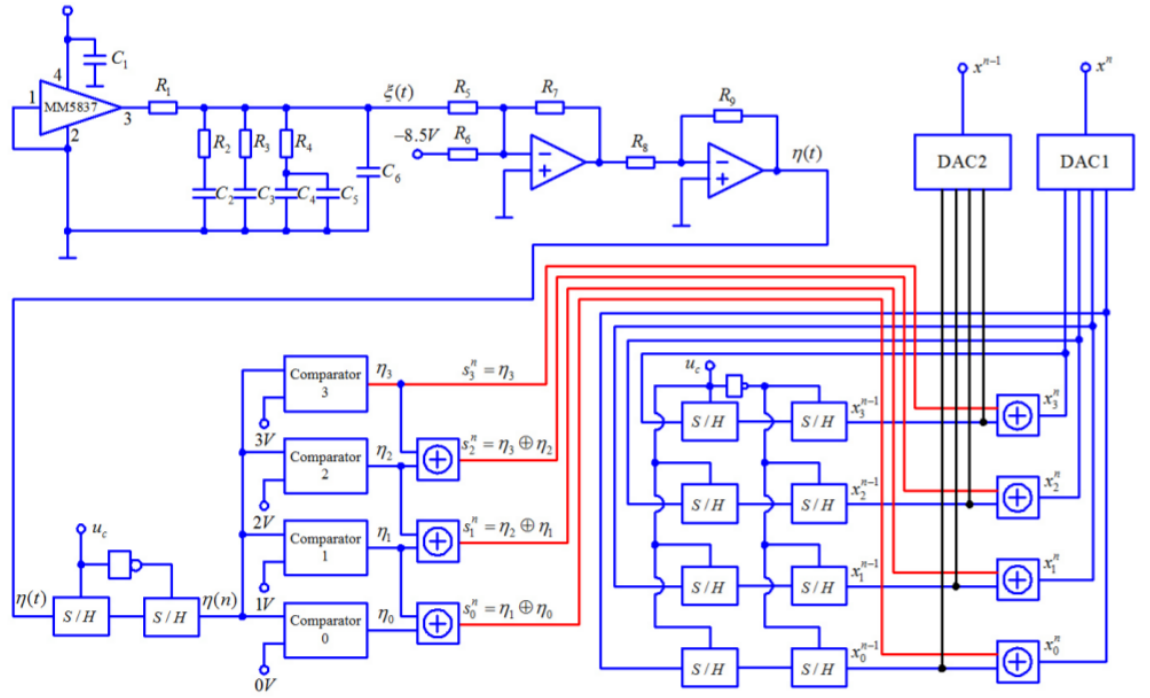


Figure 9: The general IDCS circuit

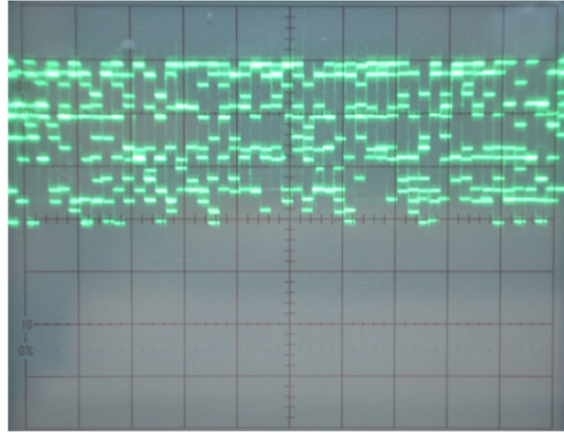


Figure 10: The experimental observations of IDCS

expression as follows:

$$\begin{cases} x_3^n = x_3^{n-1} \oplus s_3^n \\ x_2^n = x_2^{n-1} \oplus s_2^n \\ x_1^n = x_1^{n-1} \oplus s_1^n \\ x_0^n = x_0^{n-1} \oplus s_0^n \end{cases} \quad (20)$$

which totally corresponds to the chaotic iterations that have been studied in the first part of this article. According to Equation 20, we obtain the corresponding circuit design iteration equation shown in Figure 7. Digital to analog converter is shown in Figure 8, where  $R_{17} = 10k\Omega$ ,  $R_{18} = 2k\Omega$ ,  $R_{19} = 60k\Omega$ , and  $R_{20} = R_{21} = 10k\Omega$ . A DAC0832 is used, it should be configured to allow the analog output  $x^n$  to continuously reflect the state of an applied digital  $D_3D_2D_1D_0 = x_3^n x_2^n x_1^n x_0^n$  on Flow-Through Operation. The logic relationship is: when the input is  $x_3^n x_2^n x_1^n x_0^n = 0000$ , the output is  $x^n = 0V$ ; when the input is  $x_3^n x_2^n x_1^n x_0^n = 0001$ , the output is  $x^n = 1V$ ; ...; when the input is  $x_3^n x_2^n x_1^n x_0^n = 1111$ , the output is  $x^n = 15V$ . The above correspondence can be adjust by the resistance  $R_{19}$ . Based on Figure 2-Figure 8, the whole basic IDCS circuit design is shown in Figure 9, with experimental observations of IDCS as shown in Figure 10, respectively.

## 5 Conclusions

In order to solve degradation of chaotic dynamic properties by finite precision effect in traditional RDCS, a novel approach for generating IDCS and its proof of the existence of chaos according to Devaney's definition is presented. We then focus on the design and circuit implementation of IDCS, with theoretical background and practical details presented together for the first time. By following these directions, hardware realization method will be further developed in the information hiding field of applications for IDCS.

## Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant 61172023; by the Specialized Research Foundation of Doctoral Subjects of Chinese Education Ministry under Grant 20114420110003; and by China Postdoctoral Science Foundation No.2014M552175.

## References

- [1] Jacques Bahi, Jean-François Couchot, Christophe Guyeux, and Qianxue Wang. Class of trustworthy pseudo random number generators. In *INTERNET 2011, the 3-rd Int. Conf. on Evolving Internet*, pages 72–77, Luxembourg, Luxembourg, June 2011.
- [2] Jacques Bahi, Xiaole Fang, Christophe Guyeux, and Qianxue Wang. Evaluating quality of chaotic pseudo-random generators. application to information hiding. *IJAS, International Journal On Advances in Security*, 4(1-2):118–130, 2011.
- [3] Jacques Bahi, Xiaole Fang, Christophe Guyeux, and Qianxue Wang. Suitability of chaotic iterations schemes using XORshift for security applications. *JNCA, Journal of Network and Computer Applications*, \*(\*)::\*\*\*-\*\*\*, 2013. Accepted manuscript. To appear.

- [4] Jacques M. Bahi and Christophe Guyeux. Topological chaos and chaotic iterations, application to hash functions. In *WCCI'10, IEEE World Congress on Computational Intelligence*, pages 1–7, Barcelona, Spain, July 2010. Best paper award.
- [5] J. Banks, J. Brooks, G. Cairns, and P. Stacey. On Devaney’s definition of chaos. *Amer. Math. Monthly*, 99:332–334, 1992.
- [6] Guanrong Chen and Dejian Lai. Making a dynamical system chaotic: feedback control of lyapunov exponents for discrete-time dynamical systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 44(3):250–253, 1997.
- [7] Guanrong Chen and Tetsushi Ueta. Yet another chaotic attractor. *International Journal of Bifurcation and chaos*, 9(07):1465–1466, 1999.
- [8] L. Chua, M. Komuro, and T. Matsumoto. The double scroll family. *IEEE Transactions on Circuits and Systems*, 33(11):1072–1118, Nov 1986.
- [9] F. Dachsel and W. Schwarz. Chaos and cryptography. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 48(12):1498–1509, Dec 2001.
- [10] Robert L. Devaney. *An Introduction to Chaotic Dynamical Systems*. Addison-Wesley, Redwood City, CA, 2nd edition, 1989.
- [11] G. Heidari-Bateni and C. D. McGillem. A chaotic direct-sequence spread-spectrum communication system. *IEEE Transactions on Communications*, 42(234):1524–1527, Feb 1994.
- [12] SJ Li, G Chen, and XQ Mou. On the dynamical degradation of digital piecewise linear chaotic maps. *International Journal of Bifurcation and Chaos*, 15(10):3119 – 3151, 2005. This is an electronic version of an article published as *International Journal of Bifurcation and Chaos*, Volume 15, Issue 10, 2005, pp 3119–3151 DOI:10.1142/S0218127405014052 © World Scientific Publishing Company. <http://www.worldscinet.com/ijbc/ijbc.shtml>.
- [13] T. Lin and L. O. Chua. On chaos of digital filters in the real world. *IEEE Transactions on Circuits and Systems*, 38(5):557–558, May 1991.
- [14] Edward N. Lorenz. Deterministic Nonperiodic Flow. *J. Atmos. Sci.*, 20(2):130–141, March 1963.
- [15] Robert M. May. Simple mathematical models with very complicated dynamics. *Nature*, 261(5560):459–467, jun 1976.
- [16] T Sang, R. L. Wang, and Y. X. Yan. Perturbance-based algorithm to expand cycle length of chaotic key stream. *Electronics Letters*, 34:873–874, 1998.



- [17] Fan ZHENG, Xiao-jian TIAN, Jing-yi SONG, and Xue-yan LI. Pseudo-random sequence generator based on the generalized henon map. *The Journal of China Universities of Posts and Telecommunications*, 15(3):64–68, 2008.